Polarization and visibility of higher-order rainbows

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The degree of polarization of rainbows of order k with $k \geq 3$ is bounded in the interval [75%, 78%], where 75% is the limit for $k \to \infty$. A polarization filter can improve the signal-to-background ratio of the third and fourth rainbows by a factor of 2, which may lift their visibilities in natural circumstances above the threshold of human visual perception. Under optimal circumstances, the latter may be true for the recently photographed green fingerprint of the fifth rainbow, even without the aid of a polarization filter. The prospects for observing the sixth rainbow are unclear. There exists a possibility that the signal of the natural seventh rainbow (appearing at 64° from the Sun) may be separated from its background if photographed under perfect conditions through a polarization filter.

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1. Introduction

With the photographic detection in May 2011 of a natural third rainbow [1], one month afterward of a fourth rainbow [2], and in 2012 of a fifth rainbow [3,4], the study of higher-order rainbows has changed from a theoretical [5] or laboratory [6,7] issue to a topic related to features that occurs in real nature. This inspires us to study the polarization of higher-order rainbows from a generic standpoint, and then to focus on the effect of the use of a polarization filter by chasing these rainbows in nature. Our study seems to indicate that the use of a polarization filter may promote the fourth and fifth rainbows from photographic objects into features visible by the human eye. It also hints at the possibility that the seventh rainbow may become visible in an image-processed photograph taken through a polarization filter.

2. Polarization Formulas in Geometric Optics

We derive for rainbows of all orders k the Fresnel coefficients $F_{1,2}$ for reflection for the Descartes rainbow ray. Following [8], the expressions in this section are given as a function of p rather than k, where

$$k = p - 1. \tag{1}$$

The physical meaning of k is the number of internal reflections suffered by the rainbow ray on its path sun–drop–observer. The meaning of p is the number of chords inside a drop traced out by the transferring light ray or, equivalently, the number of hits of the ray at the wall of the drop after entrance.

The rainbow condition reads

$$\cos(i_{Desc}) = \sqrt{\frac{n^2 - 1}{p^2 - 1}}, \tag{2}$$

where i_{Desc} is the angle of incidence of the Descartes ray. Rainbows exists for $p > n$. From simple substitution in Eq. (2) one finds, for all n and p,

$$F_1 \equiv -\left(\frac{\sin(i - r)}{\sin(i + r)}\right)_{i=i_{Desc}} = \frac{1 - p}{1 + p}, \tag{3}$$

$$F_2 \equiv \left(\frac{\tan(i - r)}{\tan(i + r)}\right)_{i=i_{Desc}} = \frac{n^2 - p}{n^2 + p}, \tag{4}$$

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in which the indices 1 and 2 refer to the polarization perpendicular and parallel to the scattering plane, respectively. Then, the ratio of the polarized radiances I_1/I_2 of the kth rainbow reads

$$\frac{I_1}{I_2}(k) = \left(\frac{1-F_1^2}{1-F_2^2}\right)^2 \left(\frac{F_1^2}{F_2^2}\right)^k = \frac{1}{n^4}\left(\frac{1-p}{n^2-p}\right)^{2(p-1)}\left(\frac{n^2+p}{1+p}\right)^{2(p+1)} \quad (5)$$

with $k = p - 1$. With Eq. (5), the degree of polarization $P(k)$

$$P(k) = \frac{I_1 - I_2}{I_1 + I_2} \quad (6)$$

behaves as follows: it starts at from zero at $k = n - 1$, reaches 100% for $k = n^2 - 1$, and then decreases monotonically with k toward an asymptotic value:

$$\lim_{k\to\infty} P = \frac{\exp(4n^2 - 4) - n^4}{\exp(4n^2 - 4) + n^4}. \quad (7)$$

As the existence of a k-order rainbow requires that $k + 1 < n$, $P(k)$ is always positive. This implies that all rainbows have the same direction of polarization, namely perpendicular to the scattering plane.

Figure 1 shows the polarization of rainbows as a function of k for water. The asymptotic value for $k \to \infty$ is 75.3%, implying a high polarization for all water rainbows.

3. Improving the Signal-to-Background Ratio of Higher-Order Rainbows with a Polarization Filter

A polarization filter may help to increase the visibility of a rainbow—that is, to improve the signal-to-background ratio S/B of the rainbow. Let $I \equiv I_1 + I_2$ be the radiance without a polarization filter with polarized components $I_{1,2}$, and let $I(rainbow)$ and $I(backgr)$ refer to the kth rainbow radiance and its background, respectively. Then the change in S/B by using a polarization filter set to maximally transmitting light perpendicularly polarized to the scattering plane *(pol1)* and hence maximally transmitting the rainbow light is

$$\frac{S/B(pol1)}{S/B(unpol)} = \frac{I_1(rainbow)}{I(rainbow)} \times \frac{I(backgr)}{I_1(backgr)} = \frac{1 + P(rainbow)}{1 + P(backgr)}, \quad (8)$$

where *unpol* refers to the observation without a polarization filter. Similarly, for a polarization filter parallel to the scattering plane *(pol2)* one has

$$\frac{S/B(pol2)}{S/B(unpol)} = \frac{1 - P(rainbow)}{1 - P(backgr)}. \quad (9)$$

Equations (8) and (9) make it clear that S/B is insensitive to the use of a polarization filter when the rainbow and background have the same direction and degree of polarization. Adopting for the moment a typical value of the degree of polarization of a higher-order rainbow of +75% (Fig. 1), and assuming the background be unpolarized, S/B improves by a factor of 1.75 by a polarization filter maximally transmitting the rainbow light [Eq. (8)]. On the other hand, if the background is strongly polarized with the same direction as the rainbow, S/B can be increased by extinguishing both the rainbow and the background with the polarization filter [Eq. (9)]—a situation Rösch [9] had hinted at. However, to achieve an improvement of, say, a factor of 1.75 in S/B in this way, the background polarization should be at least +85%.

As the signal of a higher-order rainbow (that is, $k \geq 3$, or equivalently, $p \geq 4$) is in all cases much lower than the scattering by all paths with $p \leq 3$, we consider the radiance integrated over $p = 0$–3 as the background for higher-order rainbows caused by single scattering by drops. This definition implies that signals of the first- and second-order rainbows are regarded as background for a kth-order rainbow that happens to appear in the same scattering angle range as the first or second rainbow.

Figure 2 shows the polarization of the background of higher-order rainbows. The horizontal line indicates a background polarization of 75%, which is the value at which S/B is insensitive for the use of a polarization filter. In regions of the sky where the polarization is higher than 75%, the S/B ratio can be improved with the aid of a polarization filter that maximally extinguishes the rainbow light. This applies to the 25°-wide scattering interval 77°–103°, as well as to two narrow regions near the first and second rainbow angles. For higher-order rainbows appearing in all other parts of the sky, S/B can be improved with a polarization filter that maximally transmits the light of a rainbow.

Fig. 1. Rainbow polarization as a function of the rainbow order k for $n = 1.3333$. The dots are the integer values of k. Complete polarization occurs when $k = n^2 - 1$. The polarization of the seventh-order rainbow (75.9%) is 0.6% above the rainbow asymptotic value ($k \to$ infinity) of 75.3%.

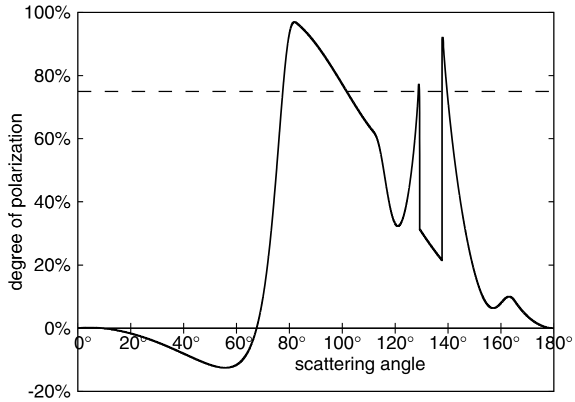

Fig. 2. Degree of polarization of singly scattered light by drops according to geometrical optics, as calculated with Laven's MiePlot program [10]. Only paths with $p \leq 3$ are considered; solar disc smearing is taken into account. The plot is for $n = 1.3333$, corresponding to a water/air index of refraction at $\lambda = 600$ nm and $T = 15°C$. A positive degree of polarization means that the plane of polarization is perpendicular to the scattering plane. The dashed line indicates the degree of polarization where the signal-to-background ratio of a higher-order rainbow is unaffected by the use of a polarization filter.

Table 1 shows the improvement factor that can be achieved on the signal-to-background ratio of rainbows up to $k = 7$ with a polarization filter [according to Eq. (8)]. As the eighth rainbow appears in the proximity of the sun (at ~20° distance) and rainbows of even higher orders become progressively weak, the table stops at the seventh rainbow. No backgrounds are considered other than those originating from single scattering by the same drops that produce the rainbow. The special case of the fifth rainbow, whose red part is in the region of the second rainbow ($p = 3$ scattering) and whose green/blue part is in Alexander's band ($p = 0$ scattering), is split accordingly into two rows.

4. Signal to Background

Table 2 presents the signal-to-background ratio in monochromatic light for higher-order rainbows up to order 7 when viewed with the naked eye and with a polarization filter that maximally transmits the rainbow light. The calculations are for a monodisperse drop size of radius 0.5 mm and solar disc diameter 0.5°, and refer to the top of main maximum of the rainbows. The 0.5 mm radius is a typical value for the mode of a rain drop spectrum during heavy rainfall [5]. All values in Table 2 have been calculated by using the Debye application of Laven's MiePlot program [10], which is expected to give more accurate results than geometric optics. The background radiance B_{drop} only takes into account single scattering by the rainbow-generating drops, in which all paths $p = 0$–3 are considered. With the exception of the fifth rainbow, all values are calculated for $n = 1.3333$, which is the index of refraction of water with respect to air for temperature 15 °C and wavelength $\lambda = 600$ nm. As before, for the fifth rainbow, the values are given for two refraction indices: for $n = 1.3350$ ($\lambda \cong 550$ nm), where its main maximum is in Alexander's band, and for $n = 1.3318$ ($\lambda \cong 660$ nm), where for drop radius 0.5 mm the main maxima of the fifth and the second rainbows coincide.

The S/B_{drop} values in Table 2 refer to ideal conditions; that is, the scattering raindrops stand out against a completely black background with no other background light than single scattering by the raindrops. As mentioned by the three 2011 papers about the third (and fourth) rainbows [1,2,5], only under conditions that approach this ideal one may one hope to detect a higher-order rainbow. The fact that $S/B_{drop}(unpol)$ for the first and second rainbows (not included in Table 2) is no less than 106 and 14, respectively, combined with the sometimes poor visibility of the these rainbows in nature, underscores their point.

In their study of the third rainbow, Lee and Laven [5] estimated on the basis of overcast measurements [11] the minimum value of the background radiance by a mechanism other than single scattering by drops to be 0.025 of B_{drop} at scattering angle $\theta = 40°$. This additional background radiance, denoted by [5] as $\omega(L_{\mathrm{OVC}})$, corresponds to 0.0265 if scaled with B_{drop} at the position of the top of the third rainbow. We denote this value 0.0265 by B_{LL} and observe that replacing B_{drop} by $B_{drop} + B_{LL}$ affects the S/B ratios of the third rainbow by only 3%.

Our value $S/B(unpol) = 0.045$ of the third rainbow compares well with the value of 0.032 as

Table 1. Higher-Order Rainbow Polarization and Improvement Factor for $n = 1.3333$, Corresponding to Water/Air at $\lambda = 600$ nm and $T = 15°C$[a]

Rainbow Order	Descartes Rainbow Angle (°)	Degree of Polarization (%)	Dominating Light Path of Background	Background Polarization (%)	Improvement Factor [Eq. (8)]
3	41.6	78.1	$p = 1$	−8.7	1.95
4	43.8	77.0	$p = 1$	−9.5	1.96
5[b]	127.5	76.2	$p = 3$	56.5	1.13
5[c]	129.4	76.8	$p = 0$	31.2	1.35
6	147.6	76.1	$p = 2$	26.5	1.39
7	63.8	75.9	$p = 1$	−8.2	1.92
Infinity	—	75.3	—	—	—

[a]Geometrical optics; only single scattering is considered.
[b]For the fifth-rainbow Descartes angle in the region of the second rainbow ($p = 3$ scattering; values are for $n = 1.3318$; $\lambda \cong 660$ nm).
[c]For the fifth-rainbow Descartes angle in Alexander's band (values are for $n = 1.3350$; $\lambda \cong 550$ nm).

Table 2. Rainbow Signal/Background Ratio S/B_{drop} and Improvement Factor for $n = 1.3333$ ($\lambda \cong 600$ nm), and Drop Radius of 0.5 mm, as Calculated Using Debye Series[a]

Rainbow Order (Angle[b])	B_{drop} (*unpol*)[c]	S/B_{drop} (*unpol*)	S/B_{drop} (*pol*1)	Improvement Factor [Eq. (8)]
3, (40.8°)	1	0.045	0.086	1.89
4, (44.9°)	0.74	0.022	0.041	1.87
5[d], (128.8°)	0.11	0.047	0.045	0.95
5[e], (130.5°)	0.011	0.77	0.98	1.28
6, (146.2°)	0.15	0.032	0.041	1.28
7, (62.3°)	0.15	0.012	0.021	1.78

[a] B_{drop} refers to the component of the background solely caused by single scattering by the rainbow-generating drops. Observations without a polarization filter are denoted by *unpol*; observations through a polarization filter maximally transmitting the rainbow light are denoted by *pol1*.

[b] Of the main maximum.

[c] Normalized with the value at the third rainbow.

[d] Obscured by the coinciding main maximum of the second-order rainbow (values are for $n = 1.3318$; $\lambda \cong 660$ nm).

[e] In Alexander's band (values are for $n = 1.3350$; $\lambda \cong 550$ nm).

calculated by Lee and Laven [5] for two heavy-rain drop size distributions. This justifies our choice to approximate the heavy-rain drop size distribution by a delta function at the peak of the Blanchard distribution for rainfall rate of 8 mm/hr. Note that the improvement factors in Table 2 are only slightly lower than the improvement factors in Table 1, where the values were calculated from geometrical optics instead of from Debye series.

5. Optical Visibility *Vis* Versus S/B

The S/B values in Table 2 refer to an instrumental situation—that is, observing a higher-order rainbow through a narrow-band filter whose bandwidth is so small that the effect of broadening of the rainbow radiance peaks due to the wavelength dependency of the rainbow angle θ_R can be neglected. In case of a visual observation in nature or a visual inspection of a photographic image, one looks for a colored structure of a few degrees wide against an uncolored background, of which the width depends primary on the wavelength dependency of θ_R. Therefore the incorporation of the angular dispersion $d\theta_R/d\lambda$ in the visibility model provides a better measure for the visibility of rainbows and allows for a fair intercomparison of the visibility rainbows of different order k. For this we define $Vis(k)$:

$$Vis(k) = C\frac{d\lambda}{d\theta_R(k)}\frac{S}{B}(k), \tag{10}$$

where C is a factor that normalizes the value of $Vis(3, unpol)$ for $B = B_{drop} + B_{LL}$ to unity:

$$\frac{1}{C} = \frac{d\lambda}{d\theta_R(3)}\frac{S}{B_{drop} + B_{LL}}(3, unpol), \tag{11}$$

so that the visibility is measured relative to that of the third rainbow. The factor $d\theta_R(k)/d\theta_R(3)$ that appears after combining Eqs. (10) and (11) increases almost linearly from 1 to 2 when k runs from 3 to 7.

Table 3 shows the estimated visibilities of rainbows up to the seventh order, with and without a polarization filter. The additional background B_{LL} is assumed to be unpolarized and independent of scattering angle. To show the sensitivity of Vis on B, values are shown for $B = B_{drop} + B_{LL}$, $B = B_{drop} + 3B_{LL}$, as well as $B = B_{drop} + 10B_{LL}$.

Table 3 shows for $k = 3$, 4 little dependency of Vis on B. This indicates that the values for Vis are robust. Grossmann *et al.* [1], who have taken the first photographical record of the third rainbow, report that they have visually observed a glimpse of it. Theusner [2] has made the second photographic record of the third together with the first photographic record of the fourth rainbow without having them observed visually. Although in his stacked images both rainbows appear amply above the threshold of photographic visibility, they are not discernable in the individual frames. Antipov [12], who took the second picture ever on which both the third and fourth rainbows are visible, reports that his photographing was prompted by a short moment in which he visually noticed “something” in the third-rainbow region, which was apparently a glimpse of the third rainbow. Edens [4] obtained during his third observation of the fifth rainbow (13 July 2013) also the third photograph ever in which both the third and fourth rainbows

Table 3. Rainbow Visibility *Vis* for $n = 1.3333$, where to B_{drop} an Unpolarized Background Radiance of a Multiple of B_{LL} is Added, in which B_{LL} is the Background Proposed by Lee and Laven [5][a]

Rainbow Order	$Vis(unpol)$ with $B = B_{drop} +$			$Vis(pol1)$ with $B = B_{drop} +$		
	$+1 \times B_{LL}$	$+3 \times B_{LL}$	$+10 \times B_{LL}$	$+1 \times B_{LL}$	$+3 \times B_{LL}$	$+10 \times B_{LL}$
3	**1**	0.95	0.81	1.9	1.8	1.5
4	0.37	0.35	0.28	0.70	0.65	0.52
5, obscured by second[b]	0.56	0.41	0.21	0.58	0.47	0.28
5, in Alex band[c]	3.3	1.4	0.46	5.1	2.2	0.75
6	0.34	0.26	0.14	0.45	0.37	0.22
7	0.11	0.08	0.05	0.19	0.14	0.08

[a] Observations without a polarization filter are denoted by *unpol*; observations through a polarization filter maximally transmitting rainbow light are denoted by *pol1*. Values are for drop radius of 0.5 mm (calculated using Debye series).

[b] $n = 1.3318$; $\lambda \cong 660$ nm.

[c] $n = 1.3350$; $\lambda \cong 550$ nm.

Table 4. Photographic Records of Higher Rainbows through 31 December 2013

	Observer[a]	Date (UTC)	Sun Elevation (°)	Visually Observed?	Discernable in Unprocessed Pictures?
Third rainbow	Grossman[b] [1]	15 May 2011	8	Yes	Yes [14]
	Theusner [2]	11 June 2011	11	No	No [15]
	Antipov [12,16]	22 June 2013	15	Yes	Yes
	Edens [3]	13 July 2013	9	Yes	Yes
	Jones [17]	8 Sept 2013	10	No	No
	Edens [18]	22 Sept 2013	18	No	Yes
Fourth rainbow	Theusner[b] [2]	11 June 2011	11	No	No [15]
	Antipov [12,16]	22 June 2013	15	No	No
	Edens [3]	13 July 2013	9	No	No
	Jones [19]	8 Sept 2013	10	No	No
	Edens [18]	22 Sept 2013	18	No	No
Fifth rainbow[c]	Edens [18]	28 July 2009	17	—	Yes
	Edens [4]	4 Sept 2009	8	—	Yes
	Edens [18]	8 July 2010	34	—	Yes
	Edens [18]	29 Aug 2010	4	—	No
	Edens [18]	9 Sept 2010	1	—	No
	Edens[b] [3,4]	8 Aug 2012	26	Not tried	Yes
	Edens [18]	8 Mar 2013	16	No	No
	Edens [3]	13 July 2013	9	No	No
	Edens [18]	16 Sept 2013	9	No	Yes
	Edens [18]	22 Sept 2013	18	No	No

[a]Edens' and Antipov's pictures are all taken through a polarization filter; their visual observations without.
[b]Discovery picture.
[c]Edens' pre-2012 fifth-rainbow pictures surfaced during a search in his pre-2012 digital images [4].

are seen, and reported that he did observe the third rainbow visually [3]. He added that the circumstances could have been even better than during his observation, as the sky background in the third/fourth rainbow region was relatively bright [13].

Table 4 summarizes the photographic observations of higher rainbows through 31 December 2013. The experiences of the observers of the third and fourth rainbows, combined with the fact that before 2011 the third rainbow had been visually observed four times [5], leads with $Vis(unpol)$ in Table 3 to the following tentative interpretation of Vis under perfect conditions:

— $Vis > 0.5$: the rainbow may be observed visually in nature;
— $Vis > 0.2$: the rainbow may become visible in strongly contrast-enhanced or stacked color photographs.

6. Practical Consequences

Tables 2 and 3 imply the following for the observation of higher-order rainbows:

The third rainbow (~40° from sun) appears in a bright region of the sky where the background radiance B_{drop}, dominated by $p = 1$ scattering, is slightly negatively polarized. A polarization filter increases Vis by a factor of 2. The third rainbow is occasionally above the limit of human perception; the use of a polarization filter can lead to more and unequivocal visual observations.

The fourth rainbow (~45° from sun) appears in the same region of the sky as the third rainbow, against a comparable background. A polarization filter increases Vis by a factor of 2, which lifts it to a level close to that of the third rainbow without a polarization filter. This implies that visual detection of the fourth rainbow becomes within reach. A facilitating factor is the very existence of the Theusner picture [2]: once the third rainbow is spotted, one knows exactly where to look for finding the fourth rainbow.

The fifth rainbow (~130° from sun) overlaps for long wavelengths with the second rainbow; for shorter wavelengths, it is in Alexander's band, where B_{drop} is two orders of magnitude lower than at ~40° from the sun. Despite the high values of Vis in Table 2, it has proven harder than expected to initially detect the green/blue traces of the fifth rainbow in Alexander's band [3,4]. A possible reason may be a serious underestimation of our values of B: the application of its $\theta \cong 40°$ value to the region of backscattering ($\theta > 90°$) may have resulted in too optimistic values. More likely even is that the contradiction stems from the shortcomings of a purely radiance-based prediction, if applied to situations in which color contrasts play a crucial role. We note that once recognized and knowing where to look for, the green part of the fifth rainbow becomes increasingly easier to detect. This is bolstered by the fact that Edens has reproduced the observation four times since the original detection and found five more cases while revisiting his pre-2012 digital rainbow images [4] (see Table 4). This feeds the hope that the photographically documented green fingerprint of the fifth rainbow in Alexander's band may be discernible to the naked eye—with or without the (in this case not very effective) aid of a polarization filter.

As to the longer wavelength, there seems to be little hope that the (red) signal of a natural fifth rainbow can ever be visually or photographically separated from its strongly colored $p = 3$ background; a polarization filter is here of no help.

The sixth rainbow (~145° from sun) stands out against the tail of the first rainbow ($p = 2$ scattering), which implies against a $(\sim +25\%)$ polarized background. The use of a polarization filter does not substantially increase the visibility of the bow. Given the problems mentioned above for extrapolating B to the region of backscattering, it remains to be proven whether the signal of the sixth rainbow can ever be separated from the $p = 2$ background.

The seventh rainbow (~60° from sun) appears in a slightly negatively polarized region of the sky, dominated by $p = 1$ scattering. The background radiance B_{drop} is a factor of 6 less than in the region of the of third and fourth rainbows. The application of B_{LL} for $\theta \cong 40°$ to the seventh-rainbow region $(\theta \cong 60°)$ seems realistic, which means that the values Vis in Table 3 represent a genuine comparison with the visibilities of the third and fourth rainbows. With a polarization filter Vis may increase to a value of ~0.2, which hints upon a possibility of direct photographic detection of the seventh rainbow on a photograph taken through a polarization filter during perfect conditions.

With regard to the **eighth rainbow** (~20° from the sun; $Vis(unpol) = 0.008$; improvement factor 1.7) or **higher rainbows**, there seems little chance for their detection in nature.

7. Conclusions and Recommendation

The conclusions can be summarized as follows:

- A polarization filter can be of significant help for photographically and visually detecting rainbows of order 3, 4, and 7 in nature.
- The third rainbow becomes more clearly visible to the naked eye if equipped with a polarization filter. With the aid of a polarization filter the fourth rainbow may become visible to the naked eye if the circumstances are as favorable as in [1–4]—that is, heavy rain and a dark background.
- Maximum benefit of the polarization effect requires the use of a transparent and colorless polarization filter. Looking with both eyes through a (large) filter rather than with one eye closed increases the chances for visual detection.
- The photographically detected green fingerprint of the fifth rainbow in Alexander's band may also be discernible with the naked eye, with or without the aid of a polarization filter.
- Our analysis regarding the prospects for detecting the sixth rainbow is inconclusive. A successful detection of the sixth rainbow may provide an estimate of the background B in the antisolar region under favorable conditions.
- There exists a possibility that the seventh rainbow (~60° from sun) could be directly detected in a single strongly contrast-enhanced picture taken with a photo camera equipped with a polarization filter. Alternatively, it may be detected in a differential polarization image obtained from two pictures taken with mutually orthogonally oriented polarizers.

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References

1. M. Grossmann, E. Schmidt, and A. Haussmann, "Photographic evidence for the third order rainbow," Appl. Opt. **50**, F134–F141 (2011).
2. M. Theusner, "Photographic observation of a natural fourth-order rainbow," Appl. Opt. **50**, F129–F133 (2011).
3. H. E. Edens, "Photographic observation of a natural fifth-order rainbow," presented at the 11th Conference on Light and Color in Nature, Fairbanks Alaska, 5–8 August 2013.
4. H. E. Edens, "Photographic observation of a natural fifth-order rainbow," Appl. Opt. **54**, B26–B34 (2015).
5. R. L. Lee, Jr. and P. Laven, "Visibility of natural tertiary rainbows," Appl. Opt. **50**, F152–F161 (2011).
6. F. Billet, "Mémoirs sur les dix-neuf premiers arcs-en-ciel de l'eau," Annales Scientifiques de l'École Normale Supérieure (série 1) **5**, 67–109 (1868).
7. J. Walker, "Multiple rainbows from single drops of water and other liquids," Am. J. Phys. **44**, 421–433 (1976).
8. H. C. van de Hulst, *Light Scattering by Small Particles* (Dover, 1981), p. 234.
9. S. Rösch, "Der Regenbogen in Wissenschaft und Kunst," Appl. Opt. **7**, 233–239 (1968).
10. P. Laven, "MiePlot computer program," www.philiplaven.com/mieplot.htm.
11. R. L. Lee, Jr., "Measuring overcast colors with all-sky imaging," Appl. Opt. **47**, H106–H115 (2008).
12. S. A. Antipov, http://www.atoptics.co.uk/fz954.htm.
13. H. E. Edens, *personal communication during the 11th Conference on Light and Color in Nature*, Fairbanks Alaska, 5–8 August 2013.
14. M. Grossmann, personal communication, 9 September 2013.
15. M. Theusner, personal communication, 10 September 2013.
16. S. A. Antipov, personal communication, 12 May 2014.
17. A. Jones, http://www.atoptics.co.uk/fz923.htm.
18. H. E. Edens, "Observations of the quinary rainbow," http://www.weatherscapes.com/quinary.html.
19. N. A. Lefaudeux, http://www.atoptics.co.uk/fz928.htm.