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Rainbows by elliptically deformed drops. II. The appearance of supernumeraries of high-order rainbows in rain showers

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The appearance of supernumeraries of high-order rainbows in heavy rain showers is explored for rainbows up to order five ($p = 6$). This is done by using a combination of the ray-theory-based first-order Möbius approximation for high-order rainbows with the Airy approximation of the rainbow radiance distribution. We conclude that supernumerary formation of rainbows of order three, four, and five is possible in natural rain showers. Supernumeraries of the third-order and fourth-order rainbows are preferentially formed near the bottom of these rainbows. A strategy for observing supernumeraries of high-order rainbows is proposed.

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1. INTRODUCTION

Two different situations can lead to the appearance of rainbow supernumeraries in Nature. The first and most obvious case is when the drop size distribution in rain, drizzle, or fog is strongly peaked in the region of small sizes. Then the spacing between the supernumeraries is directly related to the size of the dominant drop size, and the (weak) dependence of the spacing on scattering azimuth φ is directly related to the flattening of these drops. The second case, first recognized by Fraser [1], is when the drop size distribution is very broad and flat, as occurs in heavy showers. Then the radiance of the supernumeraries and their mutual spacing are markedly dependent on the scattering azimuth, but the spacing between the supernumeraries does not contain quantitative information about the range of drop sizes in the distribution. We concentrate on this latter case.

The mechanism behind this case is as follows: consider a $(p-1)$-order rainbow, in which light rays enter the drop, and make $p-1$ internal reflections at the surface before exiting. As the drop radius a increases, the deflection angle Θ of the principal maximum of the rainbow radiance distribution for a spherical drop decreases toward the Descartes rainbow deflection angle. However, the nonsphericity parameter ρ of the rain drops increases with a. If this increase of ρ results in an increase of the rainbow deflection angle Θ_p^D via the Möbius mechanism, and hence if the Möbius shift $\Delta\Theta_p > 0$, then the position of each relative maximum in the rainbow radiance distribution, labeled by $i = 0, 1, 2\ldots$, as a function of a will have a stationary point $a_{\mathrm{eff},i}$, and the supernumerary interference pattern belonging to $a_{\mathrm{eff},i}$ will become selectively visible. Hence the appearance of supernumeraries of a $(p-1)$-order rainbow in heavy showers requires that $\Delta\Theta_p > 0$.

So far, this mechanism has been investigated for the first-order and second-order rainbows only [1,2], as application to higher-order rainbows had seemed of minor significance by lack of credible observations [3]. This changed in 2011, when the first photographic observations of the third-order [4] and fourth-order [5] natural rainbows where published; a milestone that was followed by the detection of the fifth-order rainbow in 2012 [6], and one year later possibly the seventh-order [7] rainbow. This new situation stimulated us to investigate under what conditions the Fraser mechanism [1] should result in the appearance of supernumeraries of natural high-order rainbows.

Our study is split into two parts. In Part I, a general ray-theory-based first-order Möbius approximation for the general $(p-1)$-order rainbow was derived and its properties were investigated as a function of sun height angle h and index of refraction m [8]. In Part II here, the Möbius approximation is merged with the Airy approximation of the rainbow radiance distribution resulting in an analytical formula for the $(p-1)$-order rainbow radiance distribution for oblate spheroidally deformed raindrops, suitable for the investigation of the occurrence of supernumeraries of high-order rainbows during showers.

The body of this paper is organized as follows. In Section 2 the formulas of Airy theory are modified so as to include the first-order Möbius shift. In Section 3 formulas are obtained for the values of the drop diameter a for which the maxima in the modified Airy theory are stationary, which forms the basis of the Fraser mechanism for supernumerary formation. The variation of the angular spacing of the supernumeraries is also determined as a function of the sun height angle. Section 4 discusses the second-order Möbius term, needed for analyzing the second-order rainbow and for estimating the accuracy of the first-order approximation. In Section 5 the conditions under which supernumeraries may appear are presented and illustrated with a diagram. In Section 6 the results are discussed and a strategy is proposed for observing the supernumeraries of high-order rainbows in heavy showers. Lastly, in Section 7 we recount our major results and conclusions.

2. COMBINING AIRY THEORY WITH THE MÖBIUS SHIFT

Assuming that the Airy shift and the Möbius shift are additive for oblate spheroidally deformed drops with small ellipticity, the Airy approximation to the rainbow radiance distribution R can modeled in a number of different ways. For the development here, we incorporate the Möbius shift into the Airy approximation as [2]

$$R(a,\Theta) \propto a^{7/3}\mathrm{Ai}^2[f(a,\Theta)], \tag{1a}$$

with

$$f(a,\Theta) = -\frac{1}{h_{\mathrm{cubic}}^{1/3}}\left(\frac{2\pi a}{\lambda}\right)^{2/3}[\Theta - (\Theta_p^D + \Delta\Theta_p)], \tag{1b}$$

where Ai is the Airy function, Θ is the deflection angle, Θ_p^D is the Descartes rainbow deflection angle of the $(p-1)$-order rainbow for spherical particles, $\Delta\Theta_p$ is the Möbius shift, λ is the wavelength of the incident light, a is the volume-equivalent drop radius (called r in [8]), and h_{cubic} is the parameter in Airy's cubic wavefront, given by

$$h_{\mathrm{cubic}} = \frac{(p^2-1)}{p^2}\frac{\tan(\alpha_p)}{\cos^2(\alpha_p)}, \tag{1c}$$

in which α_p is the angle of incidence of the Descartes ray for the $(p-1)$-order rainbow for spherical particles. See Ref. [9] for a similar representation of h_{cubic} for $p=2$. The first three relative maxima of $\mathrm{Ai}^2(x)$ occur for $x_0 = -1.0188$, $x_1 = -3.2482$, $x_2 = -4.8201$, in which x_0, x_1, x_2 denote the principal maximum, the first supernumerary, and the second supernumerary, respectively. Equation (1a) ignores the smaller contribution to the rainbow radiance for unpolarized incident light that is proportional to the square of the derivative of the Airy function [10,11]. The effect of the derivative term for polarized incident light is evident in the wave theory results of Figs. 4(a) and 4(b) of [12].

The Möbius shift $\Delta\Theta_p$ of a $(p-1)$-order rainbow was derived in [8]. For $2 \le p \le 6$ it is given by

$$\Delta\Theta_2 = -16\rho\,\sin(\beta_2^D)\cos^3(\beta_2^D)\cos(\theta_2^D + 2h) + O(\rho^2), \tag{2a}$$

$$\Delta\Theta_3 = +64\rho\,\sin(\beta_3^D)\cos^3(\beta_3^D)\cos(2\beta_3^D) \times\cos(\theta_3^D + 2h) + O(\rho^2), \tag{2b}$$

$$\Delta\Theta_4 = -32\rho\,\sin(\beta_4^D)\cos^3(\beta_4^D)[6\cos^2(2\beta_4^D) - 1] \times\cos(\theta_4^D + 2h) + O(\rho^2), \tag{2c}$$

$$\Delta\Theta_5 = +64\rho\,\sin(\beta_5^D)\cos^3(\beta_5^D)\cos(2\beta_5^D)[8\cos^2(2\beta_5^D) - 3]\cos(\theta_5^D + 2h) + O(\rho^2), \tag{2d}$$

$$\Delta\Theta_6 = -48\rho\sin(\beta_6^D)\cos^3(\beta_6^D)\left[\frac{80}{3}\cos^4(2\beta_6^D) - 16\cos^2(2\beta_6^D) + 1\right] \times\cos(\theta_6^D + 2h) + O(\rho^2), \tag{2e}$$

in which θ_p^D is the Descartes rainbow scattering angle of the $(p-1)$-order rainbow for spherical particles which is confined to the interval $0 \le \theta_p^D \le \pi$, β_p^D is its angle of refraction, h is the sun height angle [not to be confused with the Airy cubic wavefront parameter h_{cubic} of Eq. (1c)], and where

$$\rho \equiv \frac{b_{\max} - b_{\min}}{b_{\max} + b_{\min}}, \tag{3}$$

parameterizes the nonsphericity of the oblate spheroid, in which $b_{\max}$ and $b_{\min}$ are the semi-major and semi-minor axes, respectively, of the spheroid.

The rainbow scattering angle θ_p^D is related to the rainbow deflection angle Θ_p^D by

$$\theta_p^D = \arccos[\cos(\Theta_p^D)]. \tag{4}$$

Equations (2a)–(2e) refer to the top of the rainbow, which is the rainbow segment closest to the zenith (see Section 3a of Ref. [8] for a discussion of this convention). For the bottom (i.e., the segment observed to be farthest from the zenith), the plus sign in front of the h has to be replaced by a minus sign. For other scattering azimuths φ, h, and ρ have to be replaced [2,8] by h' and ρ', respectively, where

$$\rho' = \rho[1 - \sin^2(\varphi)\cos^2(h)] + O(\rho^2), \tag{5a}$$

$$\tan(h') = \tan(h)/\cos(\varphi). \tag{5b}$$

For the dependence of ρ on a, we use the quadratic approximation to Green's formula [13]:

$$\rho \cong 0.050a^2, \tag{6}$$

as was done in Ref. [2], where a is given in millimeters. For the present analysis Eq. (6) is sufficiently accurate up to $a \cong 1.5$ mm [2].

For the numerical evaluation of Eqs. (1a)–(1c) for the top and bottom of natural rainbows we substitute the values of β_p^D and θ_p^D for $m = 1.333$ in Eqs. (2), for $\theta_p^D > 90°$ we replace the factor $\cos(\theta_p^D \pm 2h)$ by $-\cos[180° - (\theta_p^D \pm 2h)]$, we insert Eq. (6) into Eq. (2), and then we define the function $M\ddot{o}_p(h)$ by

$$\Delta\Theta_p = a^2 M\ddot{o}_p(h), \tag{7}$$

where $M\ddot{o}_p(h)$ is the Möbius shift (in radians) for $m = 1.333$ ($\lambda = 600$ nm) and $\rho = 0.050$ [hence $a = 1$ mm according to Eq. (6)].

Then, one obtains for the top of the rainbow for $2 \le p \le 6$:

$$M\ddot{o}_2(h) = -0.230\ \cos(137.9° + 2h) = +0.230\ \cos(42.1° - 2h), \tag{8a}$$

$$M\ddot{o}_3(h) = -0.013\ \cos(129.1° + 2h) = +0.013\ \cos(50.9° - 2h), \tag{8b}$$

$$M\ddot{o}_4(h) = +0.362\ \cos(41.7° + 2h), \tag{8c}$$

$$M\ddot{o}_5(h) = +1.90\ \cos(43.7° + 2h), \tag{8d}$$

$$M\ddot{o}_6(h) = -0.451\ \cos(128.2° + 2h) = +0.451\ \cos(51.8° - 2h). \tag{8e}$$

Similarly, for the bottom of the rainbow:

$$M\ddot{o}_2(h) = +0.230\ \cos(42.1° + 2h), \tag{9a}$$

$$M\ddot{o}_3(h) = +0.013\ \cos(50.9° + 2h), \tag{9b}$$

$$M\ddot{o}_4(h) = +0.362\ \cos(41.7° - 2h), \tag{9c}$$

$$M\ddot{o}_5(h) = +1.90\ \cos(43.7° - 2h), \tag{9d}$$

$$M\ddot{o}_6(h) = +0.451\ \cos(51.8° + 2h). \tag{9e}$$

3. STATIONARITY OF THE AIRY SHIFT WITH RESPECT TO *a*

The condition for an Airy maximum (either the principal maximum $i = 0$, or one of the supernumerary maxima $i = 1, 2, \ldots$) to appear in a broad drop distribution according to the Fraser mechanism [1] is that the deflection angle at the i Airy maximum $\Theta(a, x_i)$ has a relative minimum as a function of particle size a,

$$\left(\frac{\partial\Theta(a, x_i)}{\partial a}\right) = 0, \tag{10}$$

which can be satisfied only when $M\ddot{o}_p > 0$. The value of the deflection angle $\Theta(a, x_i)$ at this stationary point is denoted by $\Theta_{p,i}$.

The function $\Theta(a, x_i)$ can be obtained by substituting Eq. (7) in Eq. (1b) and putting $f(a, \Theta) = x_i$:

$$\Theta(a, x_i) - \Theta_p^D = M\ddot{o}_p(h) \times a^2 + x_i h_{\text{cubic}}^{1/3}\left(\frac{\lambda}{2\pi}\right)^{2/3} a^{-2/3}, \tag{11}$$

which is an explicit expression of the total Airy plus first-order Möbius shift of the deflection angle of the i radiance maximum of an oblate spheroidal drop with respect of the Descartes deflection angle for a spherical drop. By Eq. (10), one finds that the effective drop radius $a_{\text{eff},i}$ associated with the stationarity of the i Airy relative maximum is

$$a_{\text{eff},i}(x) = \left[\frac{h_{\text{cubic}}^{1/3}}{3}\left(\frac{\lambda}{2\pi}\right)^{2/3}\frac{x_i}{M\ddot{o}_p(h)}\right]^{3/8}. \tag{12}$$

Inserting $a_{\text{eff},i}$ in Eq. (12) gives the value of the deflection angle of stationarity $\Theta_{p,i}$ of the i Airy maximum of the $(p-1)$-order rainbow for oblate spheroidal drops, and hence of its position $\Delta\Theta_{p,i}$ with respect of the Descartes rainbow angle Θ_p^D:

$$\Delta\Theta_{p,i} \equiv \Theta_{p,i} - \Theta_p^D = [3^{-3/4} + 3^{1/4}]h_{\text{cubic}}^{1/4}\left(\frac{\lambda}{2\pi}\right)^{1/2} x_i^{3/4} M\ddot{o}_p^{1/4}(h). \tag{13a}$$

From this it follows that the spacing between the j and i radiance maxima is

$$\Theta_{p,j} - \Theta_{p,i} = [3^{-3/4} + 3^{1/4}]h_{\text{cubic}}^{1/4}\left(\frac{\lambda}{2\pi}\right)^{1/2}(x_j^{3/4} - x_i^{3/4})M\ddot{o}_p^{1/4}(h). \tag{13b}$$

Equation (13b) shows that the spacing of the supernumeraries depends on the sun height angle h via the $M\ddot{o}_p$ factor of Eqs. (8) and (9). As was mentioned at the start of Section 2, the combination of the Airy and Möbius shifts can be modeled in a number of different ways. A different model giving similar results was pursued in [14–16].

4. SPECIAL CASE: SECOND-ORDER MÖBIUS SHIFT OF THE SECOND-ORDER RAINBOW FOR WATER DROPS

The $p = 3$ rainbow for water requires special attention since the first-order Möbius approximation is small and changes sign at $m = \sqrt{9/5} = 1.342$ [2,8], which lies within the visible spectrum for the refractive index of water. This has two consequences. First, the refractive index dependence of the first factor of Eqs. (8b) and (9b) cannot be neglected and should be inserted. This can be achieved by expanding the expression for $\Delta\Theta_3$ of Eq. (2b) in a Taylor series about $m_0 = \sqrt{9/5} \cong 1.342$ to give

$$\Delta\Theta_3(m) = \Delta\Theta_3\left(\sqrt{9/5}\right) + \left(m - \sqrt{9/5}\right) \times \left(\frac{d\Delta\Theta_3}{d\beta_3^D}\right)_{\beta_3^D=45°}\left(\frac{d\beta_3^D}{dm}\right)_{m=\sqrt{9/5}} + O\left[\left(m - \sqrt{9/5}\right)^2\right]. \tag{14}$$

The first term in Eq. (14) is zero. In addition, the factors $(d\Delta\Theta_3/d\beta_3^D)$ and $(d\beta_3^D/dm)$ can be evaluated as $(d\Delta\Theta_3/d\beta_3^D)_{\beta_3^D=45°} = -32\rho\cos(\theta_3^D + 2h)$ and $(d\beta_3^D/dm)_{m=\sqrt{9/5}} = -(5/12)\sqrt{5}$ (see Ref. [2]). Substituting these into Eq. (2b) along with $\sqrt{9/5} = 1.342$, one obtains

$$\Delta\Theta_3(m) = +29.81 \times (m - 1.342)\cos(\theta_3 + 2h) + \mathrm{O}\left[\rho\left(m - \sqrt{9/5}\right)^2\right] + \mathrm{O}(\rho^2), \quad (15)$$

and Eq. (8b) for the top of the second-order rainbow becomes

$$M\ddot{o}_3(h) = +1.491 \times (1.342 - m)\cos(51° - 2h). \quad (16a)$$

Similarly, Eq. (9b) for the bottom of the second-order rainbow becomes

$$M\ddot{o}_3(h) = +1.491 \times (1.342 - m)\cos(51° + 2h). \quad (16b)$$

Second, the $\mathrm{O}(\rho^2)$ term in the Möbius expansion has to be considered. A formula for the top of the bow, and valid near $m \to \sqrt{9/5}$, was previously published in [2]. In accordance with Eqs. (11) and (13) of [2], it reads

$$(\Delta\Theta_3)_{\text{sec. order}} = \rho^2\left[-\frac{32}{3}\cos^2(51° - 2h) - 16\sin(102° - 4h)\right] - 29.81 \times (1.342 - m) \times \rho^2\left[10 - \frac{28}{3}\cos^2(51° - 2h) - 5\sin(102° - 4h)\right] + \mathrm{O}\left[\rho^2\left(m - \sqrt{9/5}\right)^2\right] + \mathrm{O}(\rho^3), \quad (17)$$

which may be rewritten as

$$(\Delta\Theta_3)_{\text{sec. order}} = \rho^2\{16.9\cos[4(h - 52.5°)] - 5.33\} - 29.81 \times (1.342 - m) \times \rho^2\{[5.33 + 6.84\cos[4(h - 59°)]\} + \mathrm{O}\left[\rho^2\left(m - \sqrt{9/5}\right)^2\right] + \mathrm{O}(\rho^3). \quad (18)$$

Originally [2], Eq. (17) was derived by repeating the calculation presented by Möbius [17] for the first-order rainbow, but now for one additional internal reflection. First, all terms of order ρ were evaluated, which yielded Eq. (2b). Then, all terms of order ρ^2 were collected and the result was, as in Eq. (14), expanded in a Taylor series about $m_0 = \sqrt{9/5}$ up to order $\mathrm{O}(m - \sqrt{9/5})$, yielding Eq. (17). In Eq. (17), the first and second term correspond to the first and second term of the Taylor expansion. In front of the second term the same number 29.81 appears as in Eq. (15). It corresponds to the numerical value at $m = \sqrt{9/5}$ of $\mathrm{d}[64\rho\sin(\beta_3^D)\cos^3(\beta_3^D)\cos(2\beta_3^D)]/\mathrm{d}m$. Inserting the relation $\rho = 0.05a^2$ [Eq. (6)] in Eq. (17) determines the size-dependence of $(\Delta\Theta_3)_{\text{sec. order}}$. We note that one can approximately validate the result of Eq. (17) for the shift of the second-order rainbow using the ray tracing technique of Yu *et al.* (Ref. [18], Fig. 8) for $m = 1.342$, which gives the entire Möbius correction to all orders in $\rho \geq 2$. See Refs. [19–21] for preliminary results of two such comparisons.

5. RESULTS

The appearance of supernumeraries during a natural rain shower at a particular segment of the rainbow arc requires that the segment is above the horizon and that the Möbius shift is positive. Figure 1 shows a visibility diagram for the

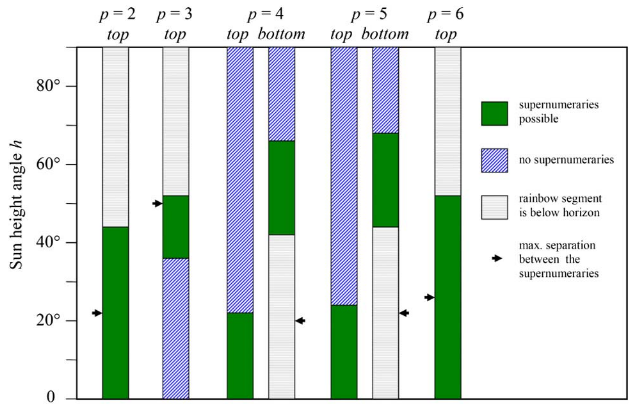

Fig. 1. Visibility diagram of supernumeraries for the top and bottom of the $(p - 1)$-order rainbow in heavy showers. Green/solid: the rainbow segment is above the horizon and supernumeraries are possible; Blue/dashed: the rainbow segment is above the horizon and supernumeraries are not possible; Grey: the rainbow segment is below the horizon. No diagrams are given for the bottom of the rainbow for $p = 2, 3, 6$ since they never raise above the horizon.

supernumeraries, and Table 1 gives some numerical results. For $p = 2$, 4–6 the calculations are based on the first-order Möbius approximation. For the second-order rainbow ($p = 3$) the points of stationarity are calculated using Eqs. (16) and (17), reproducing the results obtained in [2], including the wavelength dependence. Figure 1 and Table 1 give the result for $m = 1.3330$ and $\lambda = 600$ nm. For other realistic values of m and λ, the results are similar.

From Table 1 and Fig. 1 we note the following:

– Our model reproduces quite well the angular separation of 0.75° between the $i = 1$ and $j = 2$ supernumeraries of the first-order rainbow ($p = 2$) that was determined by Fraser [1] for $h = 0°$ and $\lambda = 550$ nm. We find from Eq. (13b) that the separation is 0.79° for $h = 0°$ and $\lambda = 550$ nm.

– Excluding the special case of the second-order rainbow ($p = 3$), one observes that the angular separation between the Descartes rainbow angle for a sphere, and the principal rainbow maxima for an oblate spheroid, and its supernumeraries increase with p (Table 1). This increase is about the same as the increase due to dispersion of these rainbows, hence the width of their colored band.

– For m in the visible range, the first-order Möbius approximation for all the rainbows we considered happens to be positive for $h = 0°$ [Eqs. (1a)–(1c)]. This implies that all rainbows considered (again with exception of $p = 3$) are capable of producing supernumeraries when the sun is at the horizon.

– Despite the inter-rainbow variation by a factor 8 ($p \neq 3$) of the constant multiplying the oscillating factor of $M\ddot{o}$, the inter-rainbow variation of a_{eff} for each rainbow maximum amounts to only a factor 1.7. The lowest value of a_{eff} found in Table 1 is 0.14 mm, and the highest is 0.42 mm. This corresponds to values of ρ between 0.001 and 0.009, and axial ratios $b_{\max}/b_{\min}$ between 1.002 and 1.018.

– For the first-order, second-order, and fifth-order rainbows ($p = 2, 3, 6$), the maximum angular separation between the supernumeraries occurs at the top of the rainbow. In contrast, for the third-order ($p = 4$) and fourth-order ($p = 5$) rainbows,

Table 1. Supernumeraries in Rain Showers with a Broad Drop Size Distribution[a]

			Principal Maximum		First Supernumerary		Second Supernumerary	
	Descartes Rainbow Scattering Angle	Sun Height Angle with Largest Separation Between Maxima	$a_{\mathrm{eff},0}$	Angular Separation from Descartes Rainbow Angle	$a_{\mathrm{eff},1}$	Angular Separation from Principal Maximum	$a_{\mathrm{eff},2}$	Angular Separation from First Supernumerary
First-order ($p = 2$)	137.9°	Rainbow top: 21° (bottom: −21°)[b]	0.14 mm	1.03°	0.22 mm	1.42°	0.25 mm	0.84°
Second-order ($p = 3$)[c]	129.1°	Rainbow top: 50° (bottom: −50°)	0.46 mm	0.78°	0.62 mm	1.21°	0.68 mm	0.76°
Third-order ($p = 4$)	41.7°	(Rainbow top: −21°) bottom 21°	0.17 mm	2.29°	0.26 mm	3.18°	0.30 mm	1.88°
Fourth-order ($p = 5$)	43.7°	(Rainbow top: −22°) bottom: 22°	0.23 mm	2.35°	0.36 mm	3.25°	0.42 mm	1.93°
Fifth-order ($p = 6$)	128.2°	Rainbow top: 26° (bottom: −26°)	0.18 mm	3.37°	0.28 mm	4.67°	0.32 mm	2.77°

[a]Values are for $m = 1.333$ and $\lambda = 600$ nm. The drop size a_{eff} giving the main contribution to the rainbow interference maxima is listed.
[b]Values with negative sun height angles are listed in parentheses.
[c]Second-order Möbius approximation [Eq. (17)] is taken into account.

this most favorable situation occurs at the bottom of the rainbow at $h \cong 20°$, when this segment is still below the horizon.

– We add to this that the angular separation between principal maximum, first supernumerary, and second supernumerary does not decrease dramatically as a function of the azimuthal angle φ. Its proportionality to $M\ddot{o}^{1/4}$ [Eq. (13b)] indicates that at 60° from top or bottom the decrease in separation is 30% at most [see Eqs. (5a) and (5b)].

6. DISCUSSION

We applied our previously obtained [8] general formula for the Möbius shift to $p \leq 6$ rainbows in an attempt to investigate whether, and if so under what conditions, supernumeraries of high-order rainbows (up to the fifth-order, hence up to $p = 6$) may show up in natural showers of a broad drop size distribution. Special focus is on the third-order and fourth-order rainbow. The main conclusion is that the highest likelihood for supernumeraries of the third-order and fourth-order rainbows ($p = 4, 5$) to appear is near the bottom of these bows rather than near their tops. This implies that a successful search for these supernumeraries should concentrate on the bottom regions of these bows and preferably take place for sun height angles larger than 35°–40°.

So far, supernumeraries of the third-order and fourth-order rainbows have not been observed in Nature. In light of the forgoing discussion, that may not be a surprise. At sun heights favorable for this type of observation, the first-order rainbow may be well below the horizon so that the observer lacks an important hint that a high-order rainbow may be present on the other side of the celestial sphere. Perhaps it is not a coincidence that all early observations of these high-order rainbows [22] took place at sun heights below 20°, when the first-order and second-order rainbows were well above the horizon.

Contrary to these other cases, a supernumerary of the fifth-order rainbow ($p = 6$) has been observed. It appeared in the discovery photograph of this rainbow, and has been extensively analyzed by Edens [6]. However, the observational data of the simultaneously appearing first- and second-order rainbows and their supernumeraries led him to conclude that the $p = 6$ supernumerary emerged because of a sharply peaked rain drop size distribution rather than by the Fraser mechanism discussed in this paper. This is consistent with Edens' report [6] that the rainbows showed up in light rain, rather than in a heavy shower, as well as with the observed angular distance to the principal maximum of $3.6° \pm 0.3°$ rather than 4.7° (Table 1). His other photographic observations of the fifth-order rainbow [22], many of which took place in heavy showers, are still waiting to be searched for supernumeraries [23].

The present analysis is based on a number of approximate formulas. For the drop flattening, we used a quadratic fit to Green's [13] relation between the (equivalent) drop radius and the nonsphericity parameter ρ, as displayed in his Table 1. At the calculated values of the effective radius of drops creating supernumeraries a_{eff} (Table 1), the quadratic approximation to Green's formula is accurate to within 0.5%, justifying its use here. The same holds for the spheroid approximation to the shape of a falling drop, which is considered to be correct for drop radii up to about 0.5 mm [24]. This is also within the range of the values of a_{eff} in our Table 1.

In this paper, the required extension of the first-order Möbius approximation to high-order rainbows is new [8,25]. For the $p \neq 3$ rainbows considered here, we believe that the first-order Möbius approximation gives sufficiently accurate results to be used for the present analysis of supernumeraries. This belief emerges from an evaluation of the $O(\rho^2)$ Möbius term for $p = 3$, which suggests that for the other rainbows, the $O(\rho^2)$ Möbius term for $a \approx 0.5$ mm is a correction to the maximum rainbow shift of order ∼1%. This accuracy is more than sufficient for our purpose. Further ray tracing computations are needed to obtain an improved assessment of the accuracy of the first-order Möbius approximation.

7. CONCLUSIONS

We have investigated for rainbows up to $p = 6$ the occurrence of supernumeraries in a broad drop-size distribution by incorporating the previously obtained [8] generalized Möbius formula for the shift of the Descartes rainbow angle into the Airy representation of the rainbow. The conclusions for the high-order rainbows ($4 \leq p \leq 6$) observed to date are:

– The third-order ($p = 4$) and fourth-order ($p = 5$) rainbows may show supernumeraries in showers. The spacing between them is roughly proportional to the dispersion of the rainbows. The most favorable location for their appearance is the rainbow segment near their bottom, which appears at solar height angles where the first-order and second-order rainbows are scarcely above the horizon, or are not above it at all.

– If the first-order and second-order rainbows are below the horizon, the observer lacks an indication that the third-order or fourth-order rainbows are ready to be photographed. A successful detection of their supernumeraries may require one to take pictures "in the blind" during heavy rainfall in the right direction.

– The fifth-order rainbow ($p = 6$) may also show its supernumeraries in showers. Their only detection thus far [6] has been for a supernumerary caused by a peaked drop size distribution instead.

– The angular distance between supernumeraries in a broad drop size distribution is markedly dependent on the scattering azimuth angle φ, but the spacing between the supernumeraries contains no information about the range of drop sizes present in the distribution. On the other hand, while the angular distance between supernumeraries in a strongly peaked drop size distribution is more weakly dependent on φ, the spacing between the supernumeraries does contain information about the dimensions of the dominant drop size.

– Only in case of the second-order rainbow the presence of supernumeraries provides direct evidence of a peaked drop size distribution in the rainbow-generating rain. In that case, the dominant drop size can straightforwardly be deduced from spacing between the supernumeraries. For all other rainbows considered here, additional information is needed to exclude that the Fraser mechanism had been at work.

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